Ants Find the Shortest Path when the Nest and Food Nodes are Connected

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- Ants deposit pheromones to help future ants navigate towards food

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- This can be modelled by a linear reinforcement model

The Model

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  - As we do, we keep track of our history, *forgetting about loops as soon as they occur*
- Once we reach F, we reinforce the weights of the edges on the $N - F$ path by 1
- We are interested in $\hat{w}^{(n)} := \frac{w^{(n)}}{n+1}$ as $n \to \infty$

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$\pi_N(\hat{w}) := \sum_{x \in V : x \sim N} \hat{w}_{Nx}^{(n)} \to 1$

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Proof.

$\pi_N(\hat{w}^{(n)}) = \frac{\sum_{x \in V : x \sim N} w_{Nx}^{(n)}}{n+1}$ □

Theorem

Theorem 2

$\hat{w}_e^{(n)} \to 1$ *and all the other weights tend to 0*

The Edge Reinforcements are Polyá Urns

What is a Polya Urn?

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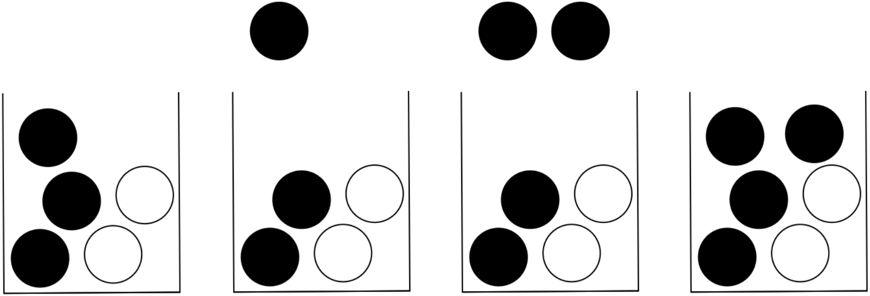
Standard Urn Model:



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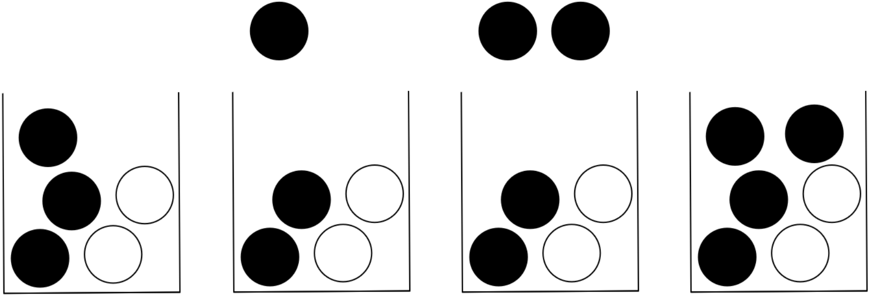

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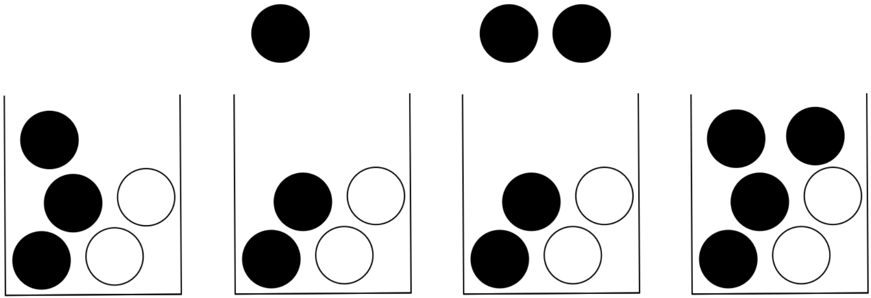

Let $w^{(n)} = \#$ white balls at time n. $(w^{(n)})_{n \geq 0}$ is an Urn process when

$$\mathbb{P}\left(w^{(n+1)} = w^{(n)} + 1 \middle| w^{(0)}, \dots, w^{(n)}\right) = \frac{\#\text{ white balls at time } n}{n+1} = \hat{w}^{(n)}$$

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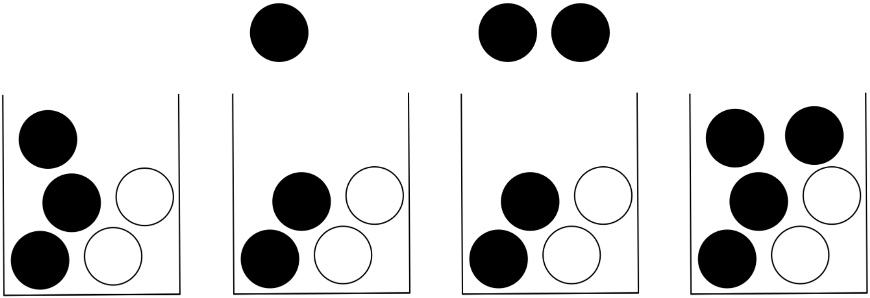

Let $w^{(n)} = \#$ white balls at time n. $(w^{(n)})_{n\geq 0}$ is a p-Urn process when

$$\mathbb{P}\left(w^{(n+1)} = w^{(n)} + 1 \middle| w^{(0)}, ..., w^{(n)}\right) = p(\hat{w}^{(n)}), p : (0,1) \mapsto (0,1)$$

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Lemma 3

If $p(\hat{w}) > \hat{w}$ for every $\hat{w} \in (0, c)$ for some $c \in (0, 1)$, then $\liminf_{n\to\infty} \hat{w}^{(n)} \geq c$

Calculating Reinforcement Probabilities

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Lemma 4

$p_x(\hat{w}) = \mathbb{P}(\tau_N^x < \tau_F^x | \hat{w}) p(\hat{w})$

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&= \frac{\hat{w}_e}{\hat{w}_e + \sum_{x \in S} \hat{w}_{Nx}} + \sum_{x \in S} \frac{\hat{w}_{Nx}}{\hat{w}_e + \sum_{x \in S} \hat{w}_{Nx}} \mathbb{P}(\tau_N^x < \tau_F^x | \hat{w}) p(\hat{w})
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$$\Rightarrow \left(\hat{w}_e + \sum_{x \in S} \hat{w}_{Nx} \right) p(\hat{w}) = \hat{w}_e + p(\hat{w}) \sum_{x \in S} \hat{w}_{Nx} \mathbb{P}(\tau_N^x < \tau_F^x | \hat{w})$$

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\Rightarrow & \, p(\hat{w}) = \frac{\hat{w}_e}{\hat{w}_e + \sum_{x \in S} \hat{w}_{Nx} \mathbb{P}(\tau_F^x < \tau_N^x | \hat{w})} = p(\hat{w}_e | \hat{w}_{-e})
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Proof of Result

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  - $\Leftrightarrow \hat{w}_e + \sum_{x \in S} \hat{w}_{Nx} \mathbb{P}(\tau_F^x < \tau_N^x | \hat{w}) < 1$
- Recall Lemma 1: $\pi_N(\hat{w}) := \hat{w}_e + \sum_{x \in S} \hat{w}_{Nx} \to 1$

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- So $\liminf \hat{w}_e \geq c$
  - But we assumed nothing about c!!!

Future Work

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Future Work

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- What about when N and F not connected?
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- Other reinforcement rules

Thank You

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$$\hat{w}_e^{(n+1)} - \hat{w}_e^{(n)} = \frac{w_e^{(n)} + \mathbb{1}_{\mathcal{A}^{(n)}}}{n+2} - \frac{w_e^{(n)}}{n+1}$$

Lemma 3 Sketch Proof

Let $\mathcal{A}^{(n)} := \left\{e \text{ reinforced} \middle| \hat{w}^{(n)}\right\}$.

$$\begin{aligned}\hat{w}_e^{(n+1)} - \hat{w}_e^{(n)} &= \frac{w_e^{(n)} + \mathbb{1}_{\mathcal{A}^{(n)}}}{n+2} - \frac{w_e^{(n)}}{n+1} \\ &= w_e^{(n)}\left(\frac{1}{n+2} - \frac{1}{n+1}\right) + \frac{1}{n+2}\mathbb{1}_{\mathcal{A}^{(n)}}\end{aligned}$$

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&= w_e^{(n)} \left(\frac{1}{n+2} - \frac{1}{n+1} \right) + \frac{1}{n+2} \mathbb{1}_{\mathcal{A}^{(n)}} \\
&= -\frac{1}{n+2} \hat{w}_e^{(n)} + \frac{1}{n+2} \left(\mathbb{1}_{\mathcal{A}^{(n)}} - p(\hat{w}_e^{(n)}) + p(\hat{w}_e^{(n)}) \right)
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&= \frac{1}{n+2} \left(p(\hat{w}_e^{(n)}) - \hat{w}_e^{(n)} + \mathbb{1}_{\mathcal{A}^{(n)}} - p(\hat{w}_e^{(n)}) \right)
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\end{aligned}$$

This makes $\left(\hat{w}_e^{(n)} \right)_{n \geq 0}$ a Stochastic Approximation Process:

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\end{aligned}
$$

This makes $\left(\hat{w}_e^{(n)}\right)_{n \geq 0}$ a Stochastic Approximation Process: behaves like the dynamical system $\dot{w} = F(w)$